Satisfiability Modulo Theories

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Disclamer: The literature on SMT and its applications is vast. The bibliographic references provided here are just a sample. Apologies to all authors whose work is not cited.

Introduction

A (Very) Brief History of Automated Reasoning

Philosophers have long dreamed of machines that can reason. The pursuit of this dream has occupied some of the best minds and led both to great acheivements and great disappointments.

~1700
Leibniz – mechanized human reasoning

1928
Hilbert

Entscheidungs-problem

1936
Church – lamda calculus

Turing – reduction halting problem

1954
Davis – decision procedure for Presburger arithmetic

Automated Reasoning: A Failure?

- At the turn of the century, automated reasoning was still considered by many to be impractical for most real-world applications
- Interesting problems appeared to be beyond the reach of automated methods because of decidability and complexity barriers
- The dream of *Hilbert*'s mechanized mathematics or *Leibniz*'s calculating machine was believed by many to be simply unattainable

The Satisfiability Revolution

Princeton, c. 2000

- *Chaff SAT solver*: orders of magnitude faster than previous SAT solvers
- *Important observation*: many real-world problems do not exhibit worst-case theoretical performance

Palo Alto, c. 2001

- Idea: combine fast new SAT solvers with decision procedures for decidable first-order theories
- *SVC*, *CVC* solvers (Stanford); *ICS*, *Yices* solvers (SRI)
- *Satisfiability Modulo Theories* (SMT) was born

SMT solvers

SMT solvers: *general-purpose* logic engines

- Given condition X, is it possible for Y to happen
- X and Y are expressed in a *rich logical language*
    - First-order logic
    - Domain-specific reasoning
        - arithmetic, arrays, bit-vectors, data types, etc.

SMT solvers are *changing the way people solve problems*

- Instead of building a *special-purpose* solver
- *Translate* into a logical formula and use an SMT solver
- Not only easier, *often better*

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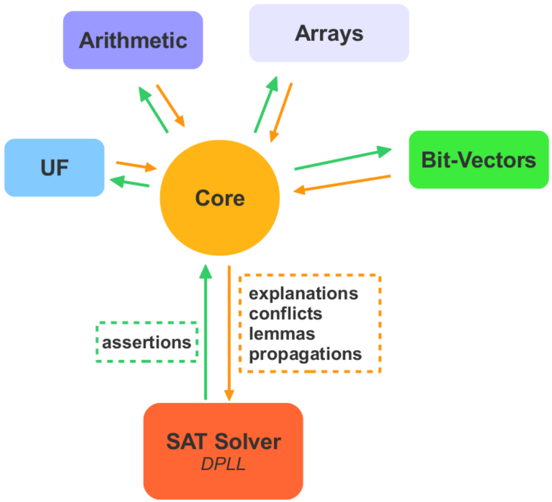
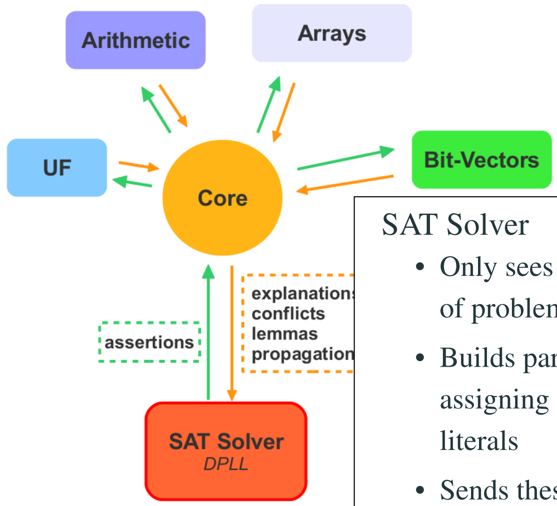
SMT Solvers


Arithmetic
Arrays
UF
Core
Bit-Vectors
assertions
explanations
conflicts
lemmas
propagations
SAT Solver
DPLL

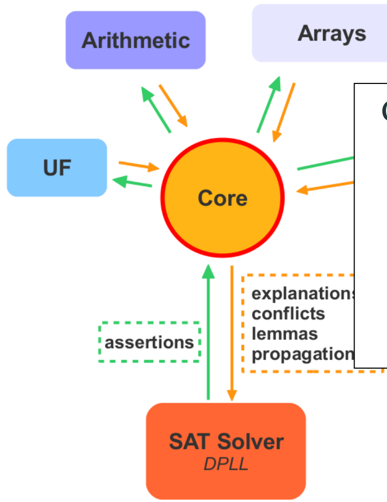
SMT Solvers



SAT Solver

- Only sees *Boolean skeleton* of problem
- Builds partial model by assigning truth values to literals
- Sends these literals to the core as *assertions*

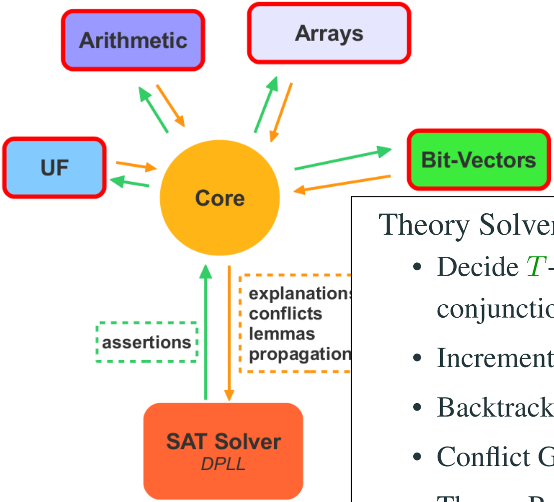
SMT Solvers



Core

- Sends each assertion to the appropriate theory
- Sends deduced literals to other theories/SAT solver
- Handles *theory combination*

SMT Solvers



Theory Solvers

- Decide T-satisfiability of a conjunction of theory literals
- Incremental
- Backtrackable
- Conflict Generation
- Theory Propagation

Theory Solvers

Theory Solvers

Given a theory T, a *Theory Solver* for T takes as input a set Φ of literals and determines whether Φ is T-satisfiable.

Φ is T-satisfiable iff there is some model M of T such that each formula in Φ holds in M.

Theories of Interest: UF

Equality ($=$) with Uninterpreted Functions [NO80, BD94, NO07]

Typically used to abstract unsupported constructs, e.g.:

- non-linear multiplication in arithmetic
- ALUs in circuits

Example: The formula

$$a * (|b| + c) = d \quad \wedge \quad b * (|a| + c) \neq d \quad \wedge \quad a = b$$

is unsatisfiable, but no arithmetic reasoning is needed:

if we abstract it to

$$mul(a, add(abs(b), c)) = d \quad \wedge \quad mul(b, add(abs(a), c)) \neq d \quad \wedge \quad a = b$$

it is still unsatisfiable

Theories of Interest: Arithmetic

Very useful, for obvious reasons

Restricted fragments (over the reals or the integers) support more efficient methods:

- Bounds: $x \bowtie k$ with $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$ [BBC+05a]
- Difference logic: $x - y \bowtie k$, with $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$ [NO05, WIGG05, CM06]
- UTVPI: $\pm x \pm y \bowtie k$, with $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$ [LM05]
- Linear arithmetic, e.g: $2x - 3y + 4z \leqslant 5$ [DdM06]
- Non-linear arithmetic, e.g: $2xy + 4xz^2 - 5y \leqslant 10$ [BLNM+09, ZM10, JdM12]

Theories of Interest: Arrays

Used in software verification and hardware verification (for memories) [SBDL01, BNO+08a, dMB09]

Two interpreted function symbols read and write

Axiomatized by:

- $\forall a\, \forall i\, \forall v.\ \mathrm{read}(\mathrm{write}(a, i, v), i) = v$
- $\forall a\, \forall i\, \forall j\, \forall v.\ i \neq j \rightarrow \mathrm{read}(\mathrm{write}(a, i, v), j) = \mathrm{read}(a, j)$

Sometimes also with *extensionality* :

- $\forall a\, \forall b.\ (\forall i.\, \mathrm{read}(a, i) = \mathrm{read}(b, i)\ \rightarrow a = b)$

Is the following set of literals satisfiable in this theory?

$$\mathrm{write}(a, i, x) \neq b,\ \mathrm{read}(b, i) = y,\ \mathrm{read}(\mathrm{write}(b, i, x), j) = y,\ a = b,\ i = j$$

Theories of Interest: Bitvectors

Useful both in hardware and software verification [BCF+07, BB09, HBJ+14]

Universe consists of (fixed-sized) vectors of bits

Different types of operations:

- *String-like*: concat, extract, ...
- *Logical*: bit-wise not, or, and, ...
- *Arithmetic*: add, subtract, multiply, ...
- *Comparison*: $<, >, \ldots$

Is this formula satisfiable over bitvectors of size 3?

$$a[1:0] \neq b[1:0] \;\wedge\; (a \mid b) = c \;\wedge\; c[0] = 0 \;\wedge\; a[1] + b[1] = 0$$

Implementing a Theory Solver: Difference Logic

We consider a simple example: difference logic.

In *difference logic*, we are interested in the satisfiability of a conjunction of arithmetic atoms.

Each atom is of the form $x - y \bowtie c$, where x and y are variables, c is a numeric constant, and $\bowtie \in \{=, <, \leqslant, >, \geqslant\}$.

The variables can range over either the *integers* (QF_IDL) or the *reals* (QF_RDL).

Difference Logic

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- $x - y = c \quad \implies \quad x - y \leqslant c \ \wedge \ x - y \geqslant c$
- $x - y \geqslant c \quad \implies \quad y - x \leqslant -c$

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- $x - y = c \quad \implies \quad x - y \leqslant c \;\wedge\; x - y \geqslant c$
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- $x - y > c \quad \implies \quad y - x < -c$

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- $x - y > c \implies y - x < -c$
- $x - y < c \implies x - y \leqslant c - 1$ (integers)

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- $x - y > c \quad \Longrightarrow \quad y - x < -c$
- $x - y < c \quad \Longrightarrow \quad x - y \leqslant c - 1$ (integers)
- $x - y < c \quad \Longrightarrow \quad x - y \leqslant c - \delta$ (reals)

Difference Logic

Now we have a conjunction of literals, all of the form $x - y \leqslant c$.

From these literals, we form a weighted directed graph with a vertex for each variable.

For each literal $x - y \leqslant c$, there is an edge $x \xrightarrow{c} y$.

The set of literals is satisfiable iff there is no cycle for which the sum of the weights on the edges is negative.

There are a number of efficient algorithms for detecting negative cycles in graphs.

Difference Logic Example

$$x - y = 5 \;\wedge\; z - y \geqslant 2 \;\wedge\; z - x > 2 \;\wedge\; w - x = 2 \;\wedge\; z - w < 0$$

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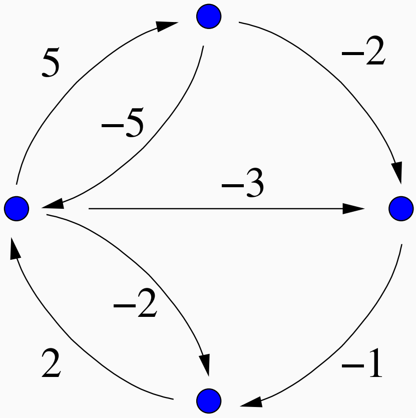
$$\begin{array}{lcl}
x - y = 5 & & x - y \leqslant 5 \wedge y - x \leqslant -5 \\
z - y \geqslant 2 & & y - z \leqslant -2 \\
z - x > 2 & \Rightarrow & x - z \leqslant -3 \\
w - x = 2 & & w - x \leqslant 2 \wedge x - w \leqslant -2 \\
z - w < 0 & & z - w \leqslant -1
\end{array}$$

Difference Logic Example



DPLL(T): Combining T-Solvers with SAT

Satisfiability Modulo a Theory T

Def. A formula is *(un)satisfiable in* a theory T, or T-*(un)satisfiable*, if there is a (no) model of T that satisfies it

Note: The T-satisfiability of quantifier-free formulas is decidable iff the T-satisfiability of conjunctions/sets of literals is decidable

(Convert the formula in DNF and check if any of its disjuncts is T-sat)

Problem: In practice, dealing with Boolean combinations of literals is as hard as in propositional logic

Solution: Exploit propositional satisfiability technology

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Lifting SAT Technology to SMT

Two main approaches:

1. "Eager" [PRSS99, SSB02, SLB03, BGV01, BV02]
   - translate into an equisatisfiable propositional formula
   - feed it to any SAT solver

Notable systems: *UCLID*

2. "Lazy" [ACG00, dMR02, BDS02, ABC$^+$02]
   - abstract the input formula to a propositional one
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   - use a theory decision procedure to refine the formula and guide the SAT solver

Notable systems: *cvc5*, *z3*

This talk will focus on the lazy approach

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Lazy Approach – Main Benefits

- Every tool does what it is good at:
  - SAT solver takes care of Boolean information
  - Theory solver takes care of theory information
- The theory solver works only with conjunctions of literals
- Modular approach:
  - SAT and theory solvers communicate via a simple API [GHN+04]
  - SMT for a new theory only requires new theory solver
  - An off-the-shelf SAT solver can be embedded in a lazy SMT system with few new lines of code (tens)

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An Abstract Framework for Lazy SMT

Several variants and enhancements of lazy SMT solvers exist

They can be modeled abstractly and declaratively as *transition systems*

A transition system is a binary relation over states, induced by a set of conditional transition rules

The framework can be first developed for SAT and then extended to lazy SMT [NOT06, KG07]

The Original DPLL Procedure

- Modern SAT solvers are based on the DPLL procedure [DP60, DLL62]
- DPLL tries to build incrementally a satisfying truth assignment M for a CNF formula F
- M is grown by
  - deducing the truth value of a literal from M and F, or
  - guessing a truth value
- If a wrong guess for a literal leads to an inconsistency, the procedure backtracks and tries the opposite value

An Abstract Framework for DPLL

States:

$$\textsf{fail} \quad \text{or} \quad \langle M, F \rangle$$

where

- M is a sequence of literals and *decision points* $\bullet$ denoting a partial truth *assignment*
- F is a set of clauses denoting a CNF *formula*

Def. If $M = M_0 \bullet M_1 \bullet \cdots \bullet M_n$ where each M_i contains no decision points

- M_i is *decision level* i of M
- $M^{[i]} \stackrel{\text{def}}{=} M_0 \bullet \cdots \bullet M_i$

An Abstract Framework for DPLL

States:

$$\textsf{fail} \quad \text{or} \quad \langle M, F \rangle$$

Initial state:

- $\langle (), F_0 \rangle$, where F_0 is to be checked for satisfiability

Expected final states:

- $\textsf{fail}$ if F_0 is unsatisfiable
- $\langle M, G \rangle$ otherwise, where
    - G is equivalent to F_0 and
    - M satisfies G

Transition Rules: Notation

States treated like records:

- M denotes the truth assignment component of current state
- F denotes the formula component of current state

Transition rules in *guarded assignment form* [KG07]

$$\frac{p_1 \quad \cdots \quad p_n}{[\mathsf{M} := e_1] \quad [\mathsf{F} := e_2]}$$

updating M, F or both when premises $p_1, \ldots, p_n$ all hold

Transition Rules for the Original DPLL

Extending the assignment

Propagate $$\frac{l_1 \vee \cdots \vee l_n \vee l \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\; l}$$

Note: When convenient, treat M as a set

Note: Clauses are treated modulo ACI of $\vee$

Decide $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$$

Note: $\mathrm{Lit}(F) \stackrel{\text{def}}{=} \{l \mid l \text{ literal of } F\} \cup \{\bar{l} \mid l \text{ literal of } F\}$

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Transition Rules for the Original DPLL

Repairing the assignment

Fail $$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \bullet \notin \mathsf{M}}{\mathsf{fail}}$$

Backtrack

$$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \mathsf{M} = M \bullet l\, N \quad \bullet \notin N}{\mathsf{M} := M\, \bar{l}}$$

Note: Last premise of **Backtrack** enforces chronological backtracking

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From DPLL to CDCL Solvers (1)

To model conflict-driven backjumping and learning, add to states a third component C whose value is either no or a *conflict clause*

States: fail or $\langle M, F, C \rangle$

Initial state:

- $\langle (), F_0, \mathsf{no} \rangle$, where F_0 is to be checked for satisfiability

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From DPLL to CDCL Solvers (2)

Replace **Backtrack** with

Conflict $\dfrac{\mathsf{C} = \mathsf{no} \quad l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M}}{\mathsf{C} := l_1 \vee \cdots \vee l_n}$

Explain $\dfrac{\mathsf{C} = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$

Backjump $\dfrac{\mathsf{C} = l_1 \vee \cdots \vee l_n \vee l \quad \mathsf{lev}\ \bar{l}_1, \ldots, \mathsf{lev}\ \bar{l}_n \leqslant i < \mathsf{lev}\ \bar{l}}{\mathsf{C} := \mathsf{no} \quad \mathsf{M} := \mathsf{M}^{[i]}\ l}$

Maintain invariant: $\mathsf{F} \models_{\mathrm{p}} \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Note: $\models_{\mathrm{p}}$ denotes propositional entailment

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Note: $l \prec_{\mathsf{M}} l'$ if l occurs before l' in M
$\mathsf{lev}\, l = i$ iff l occurs in decision level i of M

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From DPLL to CDCL Solvers (3)

Modify **Fail** to

Fail $\dfrac{C \neq \mathsf{no} \qquad \bullet \notin M}{\mathsf{fail}}$

From DPLL to CDCL Solvers (3)

Modify **Fail** to

$$\textbf{Fail}\ \frac{\mathsf{C} \neq \mathsf{no} \qquad \bullet \notin \mathsf{M}}{\mathsf{fail}}$$

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
...			

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
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$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
...			

Execution Example

$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$

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	F	no	
1	F	no	by **Propagate**
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$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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	F	no	
1	F	no	by **Propagate**
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$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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$1\ 2 \bullet 3$	F	no	by **Decide**
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$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
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$\ldots$			

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$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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$\dots$			

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
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$\ldots$			

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
...			

Execution Example

$$F := \{1,\ \ \overline{1} \lor 2,\ \ \overline{3} \lor 4,\ \ \overline{5} \lor \overline{6},\ \ \overline{1} \lor \overline{5} \lor 7,\ \ \overline{2} \lor \overline{5} \lor 6 \lor \overline{7}\}$$

M	F	C	rule
	F	no	
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \lor \overline{5} \lor 6 \lor \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \lor \overline{2} \lor \overline{5} \lor 6$	by **Explain** with $\overline{1} \lor \overline{5} \lor 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \lor \overline{2} \lor \overline{5}$	by **Explain** with $\overline{5} \lor \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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	F	no	
1	F	no	by **Propagate**
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$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

From DPLL to CDCL Solvers (4)

Also add

Learn $$\frac{\mathsf{F} \models_{\mathrm{p}} C \quad C \notin \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{C\}}$$

Forget $$\frac{\mathsf{C} = \mathsf{no} \quad \mathsf{F} = G \cup \{C\} \quad G \models_{\mathrm{p}} C}{\mathsf{F} := G}$$

Restart $$\frac{}{\mathsf{M} := \mathsf{M}^{[0]} \quad \mathsf{C} := \mathsf{no}}$$

Note: Learn can be applied to any clause stored in C when $\mathsf{C} \neq \mathsf{no}$

From SAT to SMT

Same states and transitions but

- F contains quantifier-free clauses in some theory T
- M is a sequence of theory literals and decision points
- the DPLL system is augmented with rules

  T**-Conflict**, T**-Propagate**, T**-Explain**
- maintains invariant: $\mathsf{F} \models_T \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Def. $F \models_T G$ iff every model of T that satisfies F satisfies G as well

SMT-level Rules

Fix a theory T

$$T\textbf{-Conflict}\ \frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_T \bot}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

$$T\textbf{-Propagate}\ \frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

$$T\textbf{-Explain}\ \frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Note: $\bot$ = empty clause

Note: $\models_T$ decided by theory solver

SMT-level Rules

Fix a theory T

T-Conflict $$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_T \bot}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Propagate $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

T-Explain $$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Note: $\bot$ = empty clause

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SMT-level Rules

Fix a theory T

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$$T\textbf{-Propagate}\quad \frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

$$T\textbf{-Explain}\quad \frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \dots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \dots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Note: $\bot$ = empty clause

Note: $\models_T$ decided by theory solver

Modeling a Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**, **Learn**
fail			by **Fail**

Modeling a Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**, **Learn**
fail			by **Fail**

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	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**, **Learn**
fail			by **Fail**

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	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by ***T*-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**, **Learn**
fail			by **Fail**

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M	F	C	rule
	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	$\overline{1} \lor 2 \lor 4$	by ***T*-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4$	$\overline{1} \lor 2 \lor 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4,\ \overline{1} \lor \overline{3} \lor 4$	$\overline{1} \lor \overline{3} \lor 4$	by ***T*-Conflict**, **Learn**
fail			by **Fail**

Modeling a Very Lazy Theory Approach

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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate$^+$**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by ***T*-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate$^+$**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict, Learn**
fail			by **Fail**

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	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by ***T*-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**, **Learn**
fail			by **Fail**

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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
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$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**, **Learn**
fail			by **Fail**

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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	$\overline{1} \lor 2 \lor 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4$	$\overline{1} \lor 2 \lor 4$	by **Learn**
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$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4,\ \overline{1} \lor \overline{3} \lor 4$	$\overline{1} \lor \overline{3} \lor 4$	by **T-Conflict**, **Learn**
fail			by **Fail**

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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by T**-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
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$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by T**-Conflict**, **Learn**
fail			by **Fail**

A Better Lazy Approach

The very lazy approach can be improved considerably with

- An *on-line* SAT engine,
  which can accept new input clauses on the fly
- an *incremental and explicating* T-solver,
  which can
  1. check the T-satisfiability of M as it is extended and
  2. identify a small T-unsatisfiable subset of M once M becomes T-unsatisfiable

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$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
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$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**
fail			by **Fail**

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A Better Lazy Approach

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
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fail			by **Fail**

Lazy Approach – Strategies

Ignoring **Restart** (for simplicity), a common strategy is to apply the rules using the following priorities:

1. If a clause is falsified by the current assignment M, apply **Conflict**
2. If M is T-unsatisfiable, apply T**-Conflict**
3. Apply **Fail** or **Explain+Learn+Backjump** as appropriate
4. Apply **Propagate**
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Theory Propagation

With T**-Conflict** as the only theory rule, the theory solver is used just to validate the choices of the SAT engine

With T**-Propagate** and T**-Explain**, it can also be used to guide the engine's search [Tin02]

T**-Propagate** $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

T**-Explain** $$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

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Modeling Modern Lazy SMT Solvers

At the core, current lazy SMT solvers are implementations of the transition system with rules

(1) **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump**, **Fail**

(2) T**-Conflict**, T**-Propagate**, T**-Explain**

(3) **Learn**, **Forget**, **Restart**

Reasoning by Cases in Theory Solvers

For certain theories, determining that a set M is T-unsatisfiable requires reasoning by cases.

Example: T = the theory of arrays.

$$M = \{\underbrace{r(w(a,i,x),j) \neq x}_{1},\ \underbrace{r(w(a,i,x),j) \neq r(a,j)}_{2}\}$$

$i = j$) Then, $r(w(a,i,x),j) = x$. Contradiction with 1.

$i \neq j$) Then, $r(w(a,i,x),j) = r(a,j)$. Contradiction with 2.

Conclusion: M is T-unsatisfiable

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Case Splitting

A *complete* T-solver reasons by cases via (internal) case splitting and backtracking mechanisms

An alternative is to lift case splitting and backtracking from the T-solver to the SAT engine

Basic idea: encode case splits as sets of clauses and send them as needed to the SAT engine for it to split on them [BNOT06]

Possible benefits:

- All case-splitting is coordinated by the SAT engine
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Splitting on Demand

Basic idea: encode case splits as a set of clauses and send them as needed to the SAT engine for it to split on them

Basic Scenario:

$$\mathsf{M} = \{\ldots,\ s = \underbrace{r(w(a,i,t),j)}_{s'},\ \ldots\}$$

- Main SMT module: "Is M T-unsatisfiable?"
- T-solver: "I do not know yet, but it will help me if you consider these *theory lemmas*:

$$s = s' \wedge i = j \rightarrow s = t, \quad s = s' \wedge i \neq j \rightarrow s = r(a,j)\text{"}$$

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Modeling Splitting on Demand

To model the generation of theory lemmas for case splits, add the rule

T-Learn

$$\frac{\models_T \exists \mathbf{v}(l_1 \vee \cdots \vee l_n) \quad l_1, \ldots, l_n \in L_{\mathrm{S}} \quad \mathbf{v} \text{ vars not in } \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

where L_{S} is a finite set of literals dependent on the initial set of clauses (see [BNOT06] for a formal definition of L_{S})

Note: For many theories with a theory solver, there exists an appropriate finite L_{S} for every input F

The set L_{S} does not need to be computed explicitly

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Now we can relax the requirement on the theory solver:

When $\mathsf{M} \models_{\mathrm{p}} \mathsf{F}$, *it must either*

- *determine whether* $\mathsf{M} \models_T \bot$ *or*
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The T-solver is required to determine whether $\mathsf{M} \models_T \bot$ only if all literals in L_{S} are defined in M

Note: In practice, to determine if $\mathsf{M} \models_T \bot$, the T-solver only needs a small subset of L_{S} to be defined in M

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Example — Theory of Finite Sets

$$F: \quad x = y \cup z \quad \wedge \quad y \neq \emptyset \vee x \neq z$$

M	F	rule
$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \bullet y = \emptyset$	F	by **Decide**
$x = y \cup z \bullet y = \emptyset\ x \neq z$	F	by **Propagate**
$x = y \cup z \bullet y = \emptyset\ x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by T**-Learn**
$x = y \cup z \bullet y = \emptyset\ x \neq z \bullet e \in x$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Decide**
$x = y \cup z \bullet y = \emptyset\ x \neq z \bullet e \in x\ e \notin z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Propagate**

T-solver can make the following deductions at this point:

$$e \in x \ \cdots \ \Rightarrow \ e \in y \cup z \ \cdots \ \Rightarrow \ e \in y \ \cdots \ \Rightarrow \ e \in \emptyset \ \Rightarrow \ \bot$$

This enables an application of T**-Conflict** with clause

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Example — Theory of Finite Sets

$$F: \quad x = y \cup z \quad \wedge \quad y \neq \emptyset \vee x \neq z$$

M	F	rule
$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \bullet y = \emptyset$	F	by **Decide**
$x = y \cup z \bullet y = \emptyset\ x \neq z$	F	by **Propagate**
$x = y \cup z \bullet y = \emptyset\ x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **T-Learn**
$x = y \cup z \bullet y = \emptyset\ x \neq z \bullet e \in x$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Decide**
$x = y \cup z \bullet y = \emptyset\ x \neq z \bullet e \in x\ e \notin z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Propagate**

T-solver can make the following deductions at this point:

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Applications

Some Applications of SMT

Program Analysis and Verification

- Software Model Checking[1] (e.g., BLAST, SLAM)
- K-Induction-Based Model Checking[2] (e.g., Kind)
- Concolic or Directed Automated Random Testing[3] (e.g., CUTE, KLEE, PEX)
- Program Verifiers (e.g., VCC,[4] Why3[5])
- Translation Validation for Compilers (e.g., TVOC[6])

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[1] Jhala and Majumdar, **Software Model Checking**, ACM Computing Surveys 2009.

[2] Hagen and Tinelli, **Scaling Up the Formal Verification of Lustre Programs with SMT-Based Techniques**, FMCAD'08.

[3] Godefroid, Klarlund, and Sen, **DART: Directed Automated Random Testing**, PLDI '05

[4] Dahlweid, Moskal, Santen et al. **VCC: Contract-based modular verification of concurrent C**, ICSE '09.

[5] Bobot, Filliâtre, Marché, and Paskevich, **Why3: Shepherd Your Herd of Provers**, Boogie '11.

[6] Zuck, Pnueli, Goldberg, Barrett et al., **Translation and Run-Time Validation of Loop Transformations**, FMSD '05.

Some Applications of SMT

Non-verification Applications

- AI (e.g., Robot Task Planning[7])
- Biology (e.g., Analysis of Synthetic Biology Models[8])
- Databases (e.g., Checking Preservation of Database Integrity[9])
- Network Analysis (e.g., Checking Security of OpenFlow Rules[10])
- Scheduling (e.g., Rotating Workforce Scheduling[11])
- Security (e.g., Automatic Exploit Generation[12])
- Synthesis (e.g., Symbolic Term Exploration[13])

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[7] Witsch, Skubch, et al., **Using Incomplete Satisfiability Modulo Theories to Determine Robotic Tasks**, IROS '13.

[8] Yordanov and Wintersteiger, **SMT-based analysis of Biological Computation**, NFM '13.

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[11] Erkinger, **Rotating Workforce Scheduling as Satisfiability Modulo Theories**, Master's Thesis, TU Wien, 2013.

[12] Avgerinos, Cha, Rebert et al. **Automatic Exploit Generation**, CACM '14.

[13] Kneuss, Kuraj, Kuncak, and Suter, **Synthesis Modulo Recursive Functions**, OOPSLA '13.

New Theories

SMT users are clamouring for more capabilities

New theories in the pipeline

- Theory of *sequences*
- Theory of *finite fields*
- Theory of *bags and tables*

Going forward

- There is a huge opportunity to design and implement decision procedures for new *domain-specific theories*

Scalability

Plenty of room for performance improvements

- SMT innovations continue at both the system and algorithm level
  - Example: Each year at SMT-COMP, new problems are solved that were previously too difficult for any solver
- *Parallel computing* still largely untapped

Amazon

- Ongoing collaboration with Amazon with ambitious goals for providing SMT solving as a service in the cloud
- Lots of interesting research questions about how to make use of Amazon's *massive resources* to do SMT solving on a *massive scale*

Summary

SMT solvers

- Provide *general-purpose* logical reasoning
- Can be customized for *domain-specific* reasoning
- *Enabler for formal methods*: automatic, expressive, scalable
- No shortage of *challenging research problems*
    - with immediate practical impact

More information

SMT resources

- SMT Survey Article: available at
  `http://theory.stanford.edu/~barrett/pubs/BKM14.pdf`
- SMT-LIB standards and library `http://smt-lib.org`
- SMT Competition `http://smtcomp.org`
- SMT Workshop `http://smt-workshop.org`

cvc5

- Visit the cvc5 website: `http://cvc5.github.io`
- Contact a cvc5 team member
- We welcome questions, feedback, collaboration proposals

Suggested Readings

1. R. Nieuwenhuis, A. Oliveras, and C. Tinelli. **Solving SAT and SAT Modulo Theories: From an abstract Davis-Putnam-Logemann- Loveland procedure to DPLL(T)**. Journal of the ACM, 53(6):937-977, 2006.

2. R. Sebastiani. **Lazy Satisfiability Modulo Theories**. Journal on Satisfiability, Boolean Modeling and Computation 3:141-224, 2007.

3. S. Krstić and A. Goel. **Architecting Solvers for SAT Modulo Theories: Nelson-Oppen with DPLL**. In Proceeding of the Symposium on Frontiers of Combining Systems (FroCoS'07). Volume 4720 of LNCS. Springer, 2007.

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